Fisika Matematika III

Elliptic Partial Differential Equations

- Method of Separation of Variables for Laplace Equation

Irwan Ary Dharmawan

http://phys.unpad.ac.id/jurusan/staff/dharmawan/kuliah

After this lecture …..

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

With IC and BC

Separation of Variables

$$u(x,t)??$$

Laplace Equation inside a Rectangle

$u = f_2(x)$ $\quad$ $u_1 = 0$ $\quad$ $u_2 = 0$ $\quad$ $u_3 = f_2(x)$ $\quad$ $u_4 = 0$

$u = g_2(y)$ $\quad$ $u_1 = 0$ $\quad$ $u_2 = g_2(y)$ $\quad$ $u_3 = 0$ $\quad$ $u_4 = 0$

$$\nabla^2 u = 0 \quad = \quad \nabla^2 u_1 = 0 \quad + \quad \nabla^2 u_2 = 0 \quad + \quad \nabla^2 u_3 = 0 \quad + \quad \nabla^2 u_4 = 0$$

$u = g_1(y)$ $\quad$ $u_1 = 0$ $\quad$ $u_2 = 0$ $\quad$ $u_3 = 0$ $\quad$ $u_4 = g_1(y)$

$u = f_1(x)$ $\quad$ $u_1 = f_1(x)$ $\quad$ $u_2 = 0$ $\quad$ $u_3 = 0$ $\quad$ $u_4 = 0$

Laplace Equation inside a Rectangle

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(0, y) = g_1(y)$$

$$u(L, y) = g_2(y)$$

$$u(x,0) = f_1(x)$$

$$u(x, H) = f_2(x)$$

$$\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0$$

$$u_4(0, y) = g_1(y)$$

$$u_4(L, y) = 0$$

$$u_4(x,0) = 0$$

$$u_4(x, H) = 0$$

Separation of Variables

Separation of Variables

$$u_4(x, y) = h(x)\phi(y)$$

$$h(L) = 0$$

$$\phi(0) = 0$$

$$\phi(H) = 0$$

Plug in to Laplace Eqn

$$\phi(y)\frac{d^2h}{dx^2} + h(x)\frac{d^2\phi}{dy^2} = 0$$

$$\downarrow$$

Equating

$$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2}$$

$$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = \lambda$$

Separation constant

Separation of Variables

$$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} \qquad = \lambda$$

$$\frac{d^2h}{dx^2} = \lambda h$$

$$h(L) = 0$$

$$\frac{d^2\phi}{dy^2} = -\lambda\phi$$

$$\phi(0) = 0$$

$$\phi(H) = 0$$

$$\lambda = \left(\frac{n\pi}{H}\right)^2$$

$$\phi(y) = \sin\frac{n\pi y}{H}$$

$$\frac{d^2h}{dx^2} = \left(\frac{n\pi}{H}\right)^2 h$$

$$h(x) = a_1\cosh\frac{n\pi}{H}(x-L) + a_2\sinh\frac{n\pi}{H}(x-L)$$

$$h(x) = a_2\sinh\frac{n\pi}{H}(x-L)$$

Product Solution

$$u_4(x, y) = h(x)\phi(y)$$

$$u_4(x, y) = A\sin\frac{n\pi y}{H}\sinh\frac{n\pi}{H}(x - L)$$

$$\boxed{u_4(x, y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{H}\sinh(x - L)}$$

$x = 0 \longrightarrow$

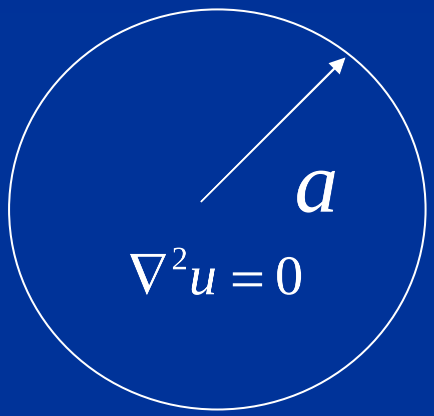
$$g_1(y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{H}\sinh\frac{n\pi}{H}(-L)$$

$$A_n \sinh\frac{n\pi}{H}(-L) = \frac{2}{H}\int_0^H g_1(y)\sin\frac{n\pi y}{H}dy$$

$$\boxed{A_n = \frac{2}{H\sinh n\pi(-L)/H}\int_0^H g_1(y)\sin\frac{n\pi y}{H}dy}$$

Laplace Equation for a Circular Disk

$$u(a,\theta) = f(\theta)$$



$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$u(a,\theta) = f(\theta)$$

Boundedness at origin $\quad |u(0,\theta)| < \infty$

periodicity

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

$$u(r,-\pi) = u(r,\pi)$$

Separation of Variables

Separation of Variables $$u(r,\theta) = \phi(\theta)G(r)$$

Plug in to Periodic BC
$$\phi(-\pi) = \phi(\pi)$$
$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

Plug in to Laplace Eqn
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}G(r)\frac{d^2\theta}{d\theta^2} = 0$$

Equating
$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda$$

Separation of Variables

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda$$

$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi$$

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 = n^2$$ Since $L = \pi$

$\sin n\theta$ $\cos n\theta$

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2G = 0$$

$$|u(0,\theta)| < \infty$$

It follows that

$$|G(0)| < \infty$$

The Solution (homework !!)

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta$$

$$0 \le r < a$$
$$-\pi < \theta \le \pi$$

$$u(a,\theta) = f(\theta)$$

$$f(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos n\theta + \sum_{n=1}^{\infty} B_n a^n \sin n\theta$$

$$-\pi < \theta \le \pi$$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$A_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta$$

$$B_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

$$n \ge 1$$